LINEAR PROGRAMMING

NOTE: 2

We will now discuss how to find solutions to a linear programming problem.

Graphical method of solving linear programming problems

Let us refer to the problem of investment in tables and chairs discussed in the previous note. We will now solve this problem graphically. Let us graph the constraints stated as linear inequalities:

$5x + y \leq 100$ ... (1)

$x + y \leq 60$ ... (2)

$x \geq 0$ ... (3)

$y \geq 0$ ... (4)

The graph of this system (shaded region) consists of the points common to all half planes determined by the inequalities (1) to (4) (Fig 12.1). Each point in this region represents a **feasible choice** open to the dealer for investing in tables and chairs. The region, therefore, is called the **feasible region** for the problem. Every point of this region is called a **feasible solution** to the problem. Thus, we have, **Feasible region** The common region determined by all the constraints including non-negative constraints x, $y \geq 0$ of a linear programming problem is called the **feasible region** (or solution region) for the problem. In Fig 12.1, the region OABC (shaded) is the feasible region for the problem. The region other than feasible region is called an **infeasible region**.

Feasible solutions: Points within and on the boundary of the feasible region represent feasible solutions of the constraints. In Fig 12.1, every point within and on the boundary of the feasible region OABC represents feasible solution to the problem.

For example, the point (10, 50) is a feasible solution of the problem and so are the points
(0, 60), (20, 0) etc. Any point outside the feasible region is called an **infeasible solution.** For example, the point (25, 40) is an infeasible solution of the problem.

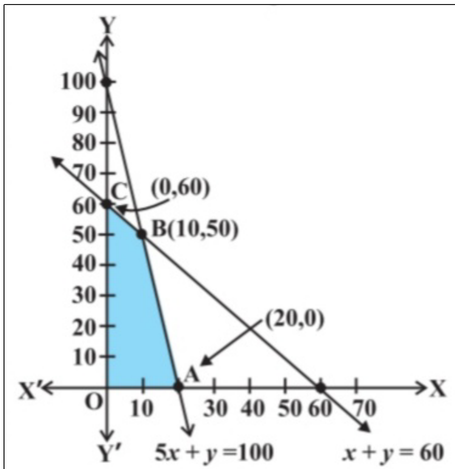


Fig 12.1

Optimal (feasible) solution: Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**

Now, we see that every point in the feasible region OABC satisfies all the constraints as given in (1) to (4), and since there are **infinitely**

many points, it is not evident how
we should go about finding a point that gives a maximum value of the objective function Z = 250*x* + 75*y.* To handle this situation, we use the following theorems which are fundamental in solving linear programming problems.

Theorem 1: Let R be the feasible region (convex polygon) for a linear programming problem and let Z = *ax* + *by* be the objective function. When Z has an optimal value (maximum or minimum), where the variables *x* and *y* are subject to constraints described by linear inequalities, this optimal value must occur at a corner point* (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let Z = *ax* + *by* be the objective function. If R is **bounded****, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.

Remark If R is **unbounded**, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (By Theorem 1).
In the above example, the corner points (vertices) of the bounded (feasible) region are: O, A, B and C and it is easy to find their coordinates as (0, 0), (20, 0), (10, 50) and (0, 60) respectively. Let us now compute the values of Z at these points. We have Vertex of the Corresponding value Feasible Region of Z (in Rs)
O (0, 0) 0
C (0, 60) 4500
B (10, 50) **6250**
A (20, 0) 5000

We observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs. This method of solving linear programming problem is referred as **Corner Point Method**. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function Z = *ax* + *by* at each corner point. Let M and *m*, respectively denote the largest and smallest values of these points.

3. (i) When the feasible region is **bounded**, M and *m* are the maximum and minimum values of Z.
(ii) In case, the feasible region is **unbounded**, we have:
4. (a) M is the maximum value of Z, if the open half plane determined by *ax* + *by* > M has no point

in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

* A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.
** A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle. Otherwise, it is called unbounded. Unbounded means that the feasible region does extend indefinitely in any direction.(very important)

We will now illustrate these steps of Corner Point Method by considering some examples:

Example 1: Solve the following linear programming problem graphically:

Maximise $Z = 4x + y$ ... (1)

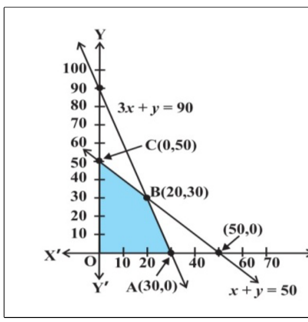
subject to the constraints:

$x + y \le 50$ ... (2)

$3x + y \le 90$ ... (3)

$x \ge 0, y \ge 0$ ... (4)

Solution: The shaded region in Fig 12.2 is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region OABC is **bounded.** So, we now use Corner Point Method to determine the maximum value of Z. The coordinates of the corner points O, A, B and C are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate Z at each corner point.



Corner Point	Corresponding value of Z
(0, 0)	0
(30, 0)	**120** ←
(20, 30)	110
(0, 50)	50

Fig 12.2

Hence, maximum value of Z is 120 at the point (30, 0).

Example 2: Solve the following linear programming problem graphically:

Minimise $Z = 200x + 500y$ ... (1)

subject to the constraints:

$x + 2y \ge 10$ ... (2)

$3x + 4y \le 24$ ... (3)

$x \geq 0$, $y \geq 0$ ... (4)

Solution The shaded region in Fig 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is **bounded**. Corner Point Corresponding value of Z

(0, 0) 0

(30, 0) **120**

(20, 30) 110

(0, 50) 50

Corner Point Corresponding value of Z

(0, 5) 2500

(4, 3) **2300**

(0, 6) 3000

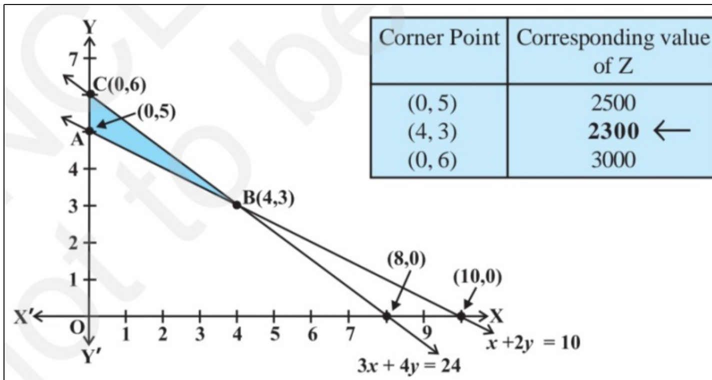


Fig 12.3

A, B and C are (0,5), (4,3) and (0,6) respectively. Now we evaluate Z = 200x + 500y at these points.

Hence, minimum value of Z is 2300 attained at the point (4, 3)

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