(1)

Determinant

A determinant is number associated to a square matrix

Def. (i) If $A = [a_{11}]_{1\times 1}$ be a one by one matrix, then det. of A or $|A| = a_{11}$

ii) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ be a 2×2 matrix

Then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called determinant of order 2

and $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$

eg. If $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, be a matrix of order 2

Then $|A| = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 3\times 6 - 5\times 4 = 18-20 = -2$

iii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ be a matrix of order 3

then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$. is called determinant of order 3 and the value of the determinant is -

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}.\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}.a_{33} - a_{32}.a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}.a_{22})$$

Minor: The minor of an element in a determinant is the determinant obtained by deleting the row and column in which the element appears.

eg in $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, minor of a_{11} in $|A|$ is $= a$

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If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, Then The minor of a_{11} of $|A|$ is

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \cdot a_{33} - a_{32} \cdot a_{23}$$

minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot a_{33} - a_{32} \cdot a_{13}$

Cofactor: Co-factor of any element of The determinant is The minor of that element multiplied by $(-1)^{i+j}$; where i and j are The no. of rows and no. of column in which The element exist.

e.g. in $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

the cofactor of a_{12} of $|A|$ is $(-1)^{1+2} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$$= (-1)^3 (a_{21} a_{33} - a_{31} a_{23})$$

$$= -(a_{21} a_{33} - a_{31} a_{23})$$

$$= a_{31} a_{23} - a_{21} a_{33}$$

Example: Find the minor and cofactor of 4 in $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ 1 & 1 & 2 \end{vmatrix}$

Solution:

Minor of $4 = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2 \times 1 - 1 \cdot (-1) = 2 + 1 = 3$

Co-factor of $4 = (-1)^{2+3} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = (-1)^5 [2 \times 1 - 1 \cdot (-1)] = -1(2+1)$

$$= -1 \times 3$$

$$= -3$$

Example: Evaluate $\begin{vmatrix} 2 & 3 & 7 \\ 2 & 7 & 2 \\ 7 & 6 & 5 \end{vmatrix}$

Solution: $\begin{vmatrix} 2 & 3 & 7 \\ 2 & 7 & 2 \\ 7 & 6 & 5 \end{vmatrix}$

$$= 2 \begin{vmatrix} 7 & 2 \\ 6 & 5 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 2 \\ 7 & 5 \end{vmatrix} + 7 \cdot \begin{vmatrix} 2 & 7 \\ 7 & 6 \end{vmatrix}$$

$$= 2 \cdot (35 - 12) - 3 \cdot (10 - 14) + 7(12 - 49)$$

$|A| = 2\cdot 23 - 3\cdot(-4) + 7(-37)$
$= 46 + 12 - 259$
$= -201$

Transpose of matrix: The matrix obtained by interchanging rows and columns of the matrix A is called the transpose of A and is denoted by A' or A^T. If A is $m\times n$, then A' is $n\times m$.

eg if $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 7 & 5 \\ 5 & 3 & 8 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 3 \\ 4 & 5 & 8 \end{bmatrix}$

Adjoint of a matrix: If $A = (a_{ij})$ is any square matrix of order $m\times n$, then transpose of the matrix (A_{ij}) where A_{ij} is co-factor of a_{ij} in det. A is called Adjoint of A and is written as Adj A. Thus if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then $\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$; find Adj. A

Solution:

Here,

$A_{11} = (-1)^{1+1}\cdot\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix}$, $A_{12} = (-1)^{1+2}\cdot\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}$, $A_{13} = (-1)^{1+3}\cdot\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix}$

$= (-1)^2\cdot(12-6)$
$= 1\cdot 6$
$= 6$

$= (-1)^3\cdot(8-6)$
$= -1\times 2$
$= -2$

$= (-1)^4\cdot(6-9)$
$= 1\cdot(-3)$
$= -3$

$A_{21} = (-1)^{2+1}\cdot\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$; $A_{22} = (-1)^{2+2}\cdot\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}$, $A_{23} = (-1)^{2+3}\cdot\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}$

$= (-1)^3\cdot(8-9)$
$= -1\cdot(-1)$
$= 1$

$= (-1)^4\cdot(4-9)$
$= 1\cdot(-5)$
$= -5$

$= (-1)^5\cdot(3-6)$
$= -1\cdot(-3)$
$= 3$

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$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$, $A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$; $A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$

$= (-1)^4 \cdot (4-9)$ $\quad = (-1)^5 \cdot (2-6)$ $\quad = (-1)^6 \cdot (3-4)$

$= 1 \cdot (-5)$ $\quad = -1 \cdot (-4)$ $\quad = 1 \cdot (-1)$

$= -5$ $\quad = 4$ $\quad = -1$

$$\therefore \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

Inverse of a Matrix:

a) **Inverse of a matrix**: If A is any n square matrix, Then n-square matrix B is called The inverse of matrix A if $AB = BA = I_n$ and is denoted by A^{-1}

b) **Singular and non-singular matrix**: The n-square matrix A is called singular matrix if $|A| = 0$ and A is called non-singular if $|A| \neq 0$

Method of determining inverse of matrix.

If A is any n-square matrix, first of all we have to find $|A|$. If $|A| \neq 0$, then we have to find out inverse of A ie A^{-1} by the following relation,

$$A^{-1} = \frac{1}{|A|} (\text{Adj. } A)$$

Symmetric matrix: A square matrix is called symmetric if $A' = A$

Skew symmetric Matrix: A square matrix is called skew symmetric if $A' = -A$.

Ex: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ if it exists.

Solution:
$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$
$$= 1 \cdot \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$
$$= 1 \cdot (12-6) - 2(4-3) + 3(2-3)$$
$$= 6 - 2 \cdot 1 + 3 \cdot (-1)$$
$$= 6 - 2 - 3$$
$$= 1 \neq 0$$

$\therefore$ A^{-1} exists

Again,

$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix}$, $A_{12} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$, $A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$

$= (-1)^2 \cdot (12-6)$ $\quad = (-1)^3 \cdot (4-3)$ $\quad = (-1)^4 (2-3)$

$= 1 \cdot 6$ $\quad = (-1) 1$ $\quad = 1 \cdot (-1)$

$= 6$ $\quad = -1$ $\quad = -1$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix}$, $A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$, $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$= (-1)^3 \cdot (8-6)$ $\quad = (-1)^4 \cdot (4-3)$ $\quad = (-1)^5 \cdot (2-2)$

$= (-1) \times 2$ $\quad = 1 \cdot 1$ $\quad = (-1) \cdot 0$

$= -2$ $\quad = 1$ $\quad = 0$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix}$, $A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$, $A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= (-1)^4 (6-9)$ $\quad = (-1)^5 \cdot (3-3)$ $\quad = (-1)^6 \cdot (3-2)$

$= 1 \cdot (-3)$ $\quad = (-1) \cdot 0$ $\quad = 1 \cdot 1$

$= -3$ $\quad = 0$ $\quad = 1$

$$\therefore \text{Adj } A = \begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T$$
$$= \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{Adj } A)$$

$$= \frac{1}{1} \cdot \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} //$$

Ex Solve by using Cramer's rule

$$3x + 5y = 17$$
$$4x - 6y = 10$$

Solution: Here, $\Delta = \begin{vmatrix} 3 & 5 \\ 4 & -6 \end{vmatrix} = 3\times(-6) - 4\times5 = -18-20 = -38$

$$\Delta_1 = \begin{vmatrix} 17 & 5 \\ 10 & -6 \end{vmatrix} = 17\times(-6) - 10\times5 = -102-50 = -152$$

$$\Delta_2 = \begin{vmatrix} 3 & 17 \\ 4 & 10 \end{vmatrix} = 3\times10 - 4\times17 = 30-68 = -38$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{\cancel{-152}^{\,4}}{\cancel{-38}} = 4$$

and $y = \frac{\Delta_2}{\Delta} = \frac{-38}{-38} = 1$

$\therefore$ Required $\left.\begin{matrix} x = 4 \\ y = 1 \end{matrix}\right\} //$

Ex: Solve the following system of equations by Cramer's rule

$$2x + y - z = 1$$
$$2x - 3y + z = 7$$
$$4y + 3z = -11$$

Solution:

$$2x + 1 \cdot y - z = 1$$
$$2x - 3y + z = 7$$
$$0 \cdot x + 4 \cdot y + 3 \cdot z = -11$$

Here, $\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 2 & -3 & 1 \\ 0 & 4 & 3 \end{vmatrix}$

$$= 2 \cdot \begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix}$$

$$= 2(-9-4) - 1(6-0) - 1(8-0)$$

$$\Delta = 2\times(-13) - 1\times 6 - 1\times 8$$
$$= -26 - 6 - 8$$
$$= -40 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & -1 \\ 7 & -3 & 1 \\ -11 & 4 & 3 \end{vmatrix}$$

$$= 1.\begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} - 1.\begin{vmatrix} 7 & 1 \\ -11 & 3 \end{vmatrix} + (-1).\begin{vmatrix} 7 & -3 \\ -11 & 4 \end{vmatrix}$$

$$= 1.(-9-4) - 1(21+11) - 1(28-33)$$
$$= -13 - 32 - 1(-5)$$
$$= -45 + 5$$
$$= -40$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 7 & 1 \\ 0 & -11 & 3 \end{vmatrix}$$

$$= 2.\begin{vmatrix} 7 & 1 \\ -11 & 3 \end{vmatrix} - 1.\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + (-1).\begin{vmatrix} 2 & 7 \\ 0 & -11 \end{vmatrix}$$

$$= 2.(21+11) - 1(6-0) - 1(-22-0)$$
$$= 64 - 6 + 22$$
$$= 86 - 6$$
$$= 80$$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 1 \\ 2 & -3 & 7 \\ 0 & 4 & -11 \end{vmatrix}$$

$$= 2.\begin{vmatrix} -3 & 7 \\ 4 & -11 \end{vmatrix} - 1.\begin{vmatrix} 2 & 7 \\ 0 & -11 \end{vmatrix} + 1.\begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix}$$

$$= 2.(33-28) - 1(-22-0) + 1.(8+0)$$
$$= 2\cdot \quad +22 + 8$$
$$= \quad +30$$
$$= \quad 40$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-40}{-40} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{80}{-40} = -\frac{80}{40} = -2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{40}{-40} = -\frac{40}{40} = -1$$

$\therefore$ Required $\left.\begin{array}{l} x=1 \\ y=-2 \\ z=-1 \end{array}\right\}$ //

Ex: An industrialist is manufacturing two types of Product A and B. M_1 and M_2 are two machines which are used for manufacturing the two types of Products. The time (in hours) taken by A and B on the machines is given below

Product	Machine M_1	M_2
A	20	10
B	10	20

If 600 hours is the time available on each machine, calculate the no. of units manufactured of each type. (G.U. 2005)

Solution: Let x and y are the units of Production of A and B respectively.

$\therefore$ A/Q,

$$20x + 10.y = 600$$
$$10x + 20y = 600$$

Here, $\Delta = \begin{vmatrix} 20 & 10 \\ 10 & 20 \end{vmatrix} = 20\times20 - 10\times10 = 400 - 100 = 300$

$\Delta_1 = \begin{vmatrix} 600 & 10 \\ 600 & 20 \end{vmatrix} = 600\times20 - 600\times10 = 12000 - 6000 = 6000$

$\Delta_2 = \begin{vmatrix} 20 & 600 \\ 10 & 600 \end{vmatrix} = 600\times20 - 10\times600 = 12000 - 6000 = 6000$

Now, $x = \frac{\Delta_1}{\Delta} = \frac{\overset{20}{60\cancel{00}}}{3\cancel{00}} = 20$

$y = \frac{\Delta_2}{\Delta} = \frac{6000}{300} = 20$

$\therefore$ The manufacturer produced 20 units of each of the products A and B.