§Concept of Limit and continuity of a function:

Limit of a function: The limit of a function $f(x)$ is constant 'l', to which the function $f(x)$ approaches as x approaches the value 'a' (a is real no.) such that the difference between the constant 'l' and the function $f(x)$ may be made as small as we please by taking x sufficiently near 'a' and is expressed as —

$$\underset{x\to a}{Lt}\ f(x) = l$$

Right hand limit: If x tends to 'a' from the right, then the limit of $f(x)$ is called the right hand limit at $x = a$ and is denoted by

$$\underset{x\to a^+}{Lt}\ f(x) \quad \text{or} \quad \underset{x\to a+0}{Lt}\ f(x).$$

Left hand limit: If x tends to 'a' from the left, then the limit of $f(x)$ is called the left hand limit at $x = a$ and is denoted by

$$\underset{x\to a^-}{Lt}\ f(x) \quad \text{or} \quad \underset{x\to a-0}{Lt}\ f(x)$$

Again, $\underset{x\to a}{Lt}\ f(x) = l$ exists, if and only if

$$\underset{x\to a^+}{Lt}\ f(x) = \underset{x\to a^-}{Lt}\ f(x) = l$$

if $\underset{x\to a^+}{Lt}\ f(x) \neq \underset{x\to a^-}{Lt}\ f(x)$, then the $\underset{x\to a}{Lt}\ f(x)$ does not exists

A few theorems on limit:

Let $f(x)$, $g(x)$ and $h(x)$ are functions of x and $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$ and $\lim_{x\to a} h(x) = C$.

Then

i) $\lim_{x\to a}\left[f(x) \pm g(x) \pm h(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \pm \lim_{x\to a} h(x)$

$= A \pm B \pm C$

ii) $\lim_{x\to a}\left[f(x)\cdot g(x)\cdot h(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \cdot \lim_{x\to a} h(x)$

$= A\cdot B\cdot C$

iii) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$ if $B \neq 0$

iv) $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n = A^n$; $(n>0)$

v) If c is a constant and $B \neq 0$ then

a) $\lim_{x\to a}\left[f(x) \pm c\right] = \lim_{x\to a} f(x) \pm c = A \pm c$

b) $\lim_{x\to a}\left[c\cdot f(x)\right] = c\cdot \lim_{x\to a} f(x) = c\cdot A$

c) $\lim_{x\to a}\left[\frac{c}{g(x)}\right] = \frac{c}{\lim_{x\to a} g(x)} = \frac{c}{B}$

Example: Find the value of $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6}$

Solution: $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6}$ [If we put $x=2$ at this stage then the function will take the form $\frac{0}{0}$, so we will not put the limiting value at this stage and hence we may proceed as follows]

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6}$$

$$= \lim_{x\to 2} \frac{x^2-x-2x+2}{x^2+3x-2x-6}$$

$$= \lim_{x\to 2} \frac{x(x-1)-2(x-1)}{x(x+3)-2(x+3)}$$

$$= \lim_{x\to 2} \frac{(x-1)\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{x-1}{x+3}$$

$$= \frac{2-1}{2+3} \quad \left[\text{Putting the limiting value } x=2\right]$$

$$= \frac{1}{5}$$ //

Alternative method.

Example: Find the value of $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6}$

Solution: Let $x = 2+h$

$\therefore x \to 2 \Rightarrow h \to 0$

$$\therefore \lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} = \lim_{h\to 0} \frac{(2+h)^2-3(2+h)+2}{(2+h)^2+(2+h)-6}$$

$$= \lim_{h\to 0} \frac{4+4h+h^2-6-3h+2}{4+4h+h^2+2+h-6}$$

$$= \lim_{h\to 0} \frac{h^2+h}{h^2+5h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+1)}{\cancel{h}(h+5)}$$

$$= \lim_{h\to 0} \frac{h+1}{h+5} \quad \left[\text{Putting } h=0\right]$$

$$= \frac{0+1}{0+5}$$

$$= \frac{1}{5}$$ //

Example - Evaluate $\lim\limits_{x\to\frac{1}{2}} \frac{2x^2+3x-2}{2x^2+5x-3}$ (G.U. 2009)

Solution:

$$\lim_{x\to\frac{1}{2}} \frac{2x^2+3x-2}{2x^2+5x-3}$$ [If we put $x=\frac{1}{2}$, the function will get the form $\frac{0}{0}$, so we proceed as follows.

$$= \lim_{x\to\frac{1}{2}} \frac{2x^2+4x-x-2}{2x^2+ x \quad x-3}$$

$$= \lim_{x\to\frac{1}{2}} \left[\frac{2x(x+2)-1(x+2)}{2x(x+3)-1(x+3)}\right]$$

$$= \lim_{x\to\frac{1}{2}} \frac{(x+2)\cancel{(2x-1)}}{(x+3)\cancel{(2x-1)}}$$

$$= \lim_{x\to\frac{1}{2}} \frac{x+2}{x+3}$$

$$= \frac{\frac{1}{2}+2}{\frac{1}{2}+3}$$

$$= \frac{5/2}{7/2}$$

$$= \frac{5}{7}$$

Example: Find $\lim\limits_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$ $(x>0)$ (G.U. 2007)

Solution: If we put $h=0$ at this stage the function will take $\frac{0}{0}$ form, so we proceed as follows

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}-\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} \quad \left[\because (a-b)(a+b) = a^2-b^2\right]$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\Rightarrow \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \;//$$

Example Find $\lim_{x\to 1} \frac{1}{\sqrt{x+1}-\sqrt{x-1}}$

Solution: If we put $x=1$ at this stage, The function will take the form $\frac{1}{0}$, so we proceed as follows =

$$\lim_{x\to 1} \frac{1}{\sqrt{x+1}-\sqrt{x-1}} = \lim_{x\to 1} \frac{\left(\sqrt{x+1}+\sqrt{x-1}\right)}{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{x+1}+\sqrt{x-1}}{\left(\sqrt{x+1}\right)^2-\left(\sqrt{x-1}\right)^2} \quad \left[(a-b)(a+b) = a^2-b^2\right]$$

$$= \lim_{x\to 1} \frac{\sqrt{x+1}+\sqrt{x-1}}{x+1-(x-1)}$$

$$= \lim_{x\to 1} \frac{\sqrt{x+1}+\sqrt{x-1}}{x+1-x+1}$$

$$= \lim_{x\to 1} \frac{\sqrt{x+1}+\sqrt{x-1}}{2}$$

$$= \frac{\sqrt{1+1}+\sqrt{1-1}}{2}$$

$$= \frac{\sqrt{2}+0}{2}$$

$$= \frac{\sqrt{2}}{2}$$

$$= \frac{1}{\sqrt{2}} \;//$$

Example A function $f(x)$ is defined as below-

$$f(x) = x^2 \; ; \; x > 1$$
$$= 2 \; , \; x = 1$$
$$= x \; ; \; x < 1$$

Does $\lim_{x \to 1} f(x)$ exist ?

Solution: The function $f(x)$ will have a limit at $x = a$ if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

Now, A/Q, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1^2 = 1$

and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

Since $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1$

$\therefore$ The limit of $f(x)$ exists at $x = 1$ and $\lim_{x \to 1} f(x) = 1$

§ Continuity of a function:

Definition: A function $f(x)$ is said to be continuous at the point $x = a$ if

i) $f(a)$ is defined i.e $f(a)$ is not in the form like $\frac{0}{0}$ and $\frac{A}{0}$ where A is constant

ii) $\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x) = f(a)$

Note: i) if $\lim_{x \to a+0} f(x) \neq \lim_{x \to a-0} f(x)$

or if $\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x) \neq f(a)$, Then The function is said to be discontinuous at $x = a$

ii) If a function $f(x)$ is continuous at all points in the interval $a < x < b$, Then the function is said to be continuous in that interval.

Example: A function is defined as follows -

$$f(x) = \begin{cases} -x & ; x < 0 \\ x & ; 0 \le x \le 1 \\ 2-x & ; x > 1 \end{cases}$$

i) Is the function is continuous at $x=0$?
ii) Is the function is continuous at $x=1$?

Solution:

i) Here,

$$\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} (-x) = 0$$

& $$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} (x) = 0$$

Again, $f(0) = 0$ $[\because f(x) = x ; 0 \le x \le 1]$

Since $\lim_{x \to 0+0} f(x) = \lim_{x \to 0-0} f(x) = f(0)$.

$\therefore$ The function is continuous at $x=0$

ii) Here,

$$\lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} (x) = 1$$

& $$\lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} (2-x) = 2-1 = 1$$

Again $f(1) = 1$ $[\because f(x) = x ; 0 \le x \le 1]$

Since $\lim_{x \to 1-0} f(x) = \lim_{x \to 1+0} f(x) = f(1)$

$\therefore$ The function is continuous at $x=1$

Example! The cost function $c(x)$ is defined by

$$c(x) = \begin{cases} 0.5x + 17.000 & ; 0 \le x \le 10,000 \\ 5x + 22,000 & ; 10.000 < x \le 20,000 \end{cases}$$

Discuss the continuity of $c(x)$ at $x = 10,000$

Solution: Here $\lim_{x \to 10.000+0} c(x) = \lim_{x \to 10.000+0} (5x+22000) = 5 \times 10000 + 22000 = 72000$

and $\lim_{x \to 10000-0} f(x) = \lim_{x \to 10000-0} (0.5x + 17000) = 0.5 \times 10000 + 17000$

$= 5000 + 17000$

$= 22000$

Again $c(10{,}000) = 0.5 \times 10000 + 17000$ $\left[\because c(x) = 0.5x + 17000 \text{ when } 0 \le x < 10{,}000\right]$
$= 5000 + 17000$
$= 22000$

Since $\lim_{x \to 1000+0} c(x) \neq \lim_{x \to 1000-0} c(x)$

$\therefore$ The function $c(x)$ is not continuous at $x = 10{,}000$

Example: The function $f(x)$ is defined as follows

$$f(x) = \begin{cases} 1-x & ; \ x < 0 \\ \sqrt{ax} & ; \ x > 0 \\ 1 & ; \ x = 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Give reason.

Solution: Here

$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} \sqrt{ax} = \sqrt{a.0} = 0$

& $\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} (1-x) = 1 - 0 = 1$

Again $f(0) = 1$ $[\because f(x) = 1, \ x = 0]$

Since $\lim_{x \to 0+0} f(x) \neq \lim_{x \to 0-0} f(x)$

$\therefore$ The function $f(x)$ is not continuous at the point $x = 0$