§ Concept of differentiation:

In differential calculus we investigate the rate of change of the dependent variable with respect to the change of the independent variable.

Increment: If the variable x changes its value from x_0 to x_1 then $(x_1 - x_0)$ is called the increment of the variable x. This increment is generally denoted by δx (read as delta x). If $x_1 > x_0$, then the increment is positive and if $x_1 < x_0$, then the increment is negative. Similarly the increment of the function $y = f(x)$ is represented by the symbol δy.

Since $\delta x = x_1 - x_0$

$$\Rightarrow x_1 = x_0 + \delta x$$

$$\therefore \delta y = f(x_1) - f(x_0)$$

$$= f(x_0 + \delta x) - f(x_0) \quad [\text{as } y = f(x)]$$

eg. if $y = 2x+1$ and x changes its value from 1 to 1.1 then $\delta x = 1.1 - 1 = 0.1$.

Again when x changes from 1 to 1.1, then the value of the dependent variable y changes from $2\times1+1 = 2+1 = 3$ to $2\times1.1+1 = 2.2+1 = 3.2$.

$$\therefore \delta y = 3.2 - 3 = 0.2$$

§ Derivative or differential co-efficient:

Let y be a function of x. Let δx be the small increment in independent variable x and δy be the corresponding increment in the dependent variable y. Suppose the ratio $\frac{\delta y}{\delta x}$ tends to a finite limit as δx tends to zero. Then this limit is called the derivative or differential coefficient of y with respect to (wrt) x and is denoted by $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Remark :

i) The differential coefficient of $f(x)$ w.r.t x at $x=a$ is denoted by $f'(a)$ or $\left[\frac{d}{dx}f(x)\right]_{x=a}$

ii) The Process of finding derivative or differential co-efficient by using definition of derivative is called "Differentiation from the First Principle".

§ Steps involved in Differentiation using The First Principle :

To find the derivative of a function the following steps are followed –

i) Assume $y = f(x)$

ii) change the value of y from y to $y+\delta y$ as x changes it value from x to $x+\delta x$, such that

$$y+\delta y = f(x+\delta x)$$

iii) Find the increement of y ie. δy by subtracting the initial value from the final value

ie $(y+\delta y)-y = f(x+\delta x)-f(x)$

$$\Rightarrow \delta y = f(x+\delta x)-f(x)$$

iv) Find the ratio $\frac{\delta y}{\delta x}$ by dividing (iii) by δx

Thus $$\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$$

v) Find the limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$

From iv) $$\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

If this limit exists and finite, then it is called the derivative of y with respect to x and it is denoted by $\frac{dy}{dx}$

Ex: Find the derivative x^2 by using the first Principle.

Sol^n: Let $y = x^2 \rightarrow$ (i)

$\therefore y + \delta y = (x+\delta x)^2 \rightarrow$ (ii)

where δx is a small change in x and δy is the corresponding change in y.

$$(ii) - (i) \Rightarrow (y+\delta y) - y = (x+\delta x)^2 - x^2$$

$$\Rightarrow \delta y = \cancel{x^2} + 2x.\delta x + (\delta x)^2 - \cancel{x^2}$$

$$\Rightarrow \delta y = 2x\,\delta x + (\delta x)^2$$

$$\Rightarrow \delta y = \delta x(2x + \delta x)$$

$$\Rightarrow \frac{\delta y}{\delta x} = 2x + \delta x$$

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x)$$

$$\Rightarrow \frac{dy}{dx} = 2x + 0$$

$$= 2x$$

Ex: Find the derivative of $\sqrt{x^2+a^2}$ by using the first Principle.

Sol^n: Let $y = \sqrt{x^2+a^2} \rightarrow$ (i)

$\therefore y + \delta y = \sqrt{(x+\delta x)^2 + a^2} \rightarrow$ (ii)

$$(ii) - (i) \Rightarrow (y+\delta y) - y = \sqrt{(x+\delta x)^2+a^2} - \sqrt{x^2+a^2}$$

$$\Rightarrow \delta y = \sqrt{(x+\delta x)^2+a^2} - \sqrt{x^2+a^2}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{(x+\delta x)^2+a^2} - \sqrt{x^2+a^2}}{\delta x}$$

$$= \frac{\left(\sqrt{(x+\delta x)^2+a^2} - \sqrt{x^2+a^2}\right)\left(\sqrt{(x+\delta x)^2+a^2} + \sqrt{x^2+a^2}\right)}{\delta x\left(\sqrt{(x+\delta x)^2+a^2} + \sqrt{x^2+a^2}\right)}$$

$$= \frac{\left\{\sqrt{(x+\delta x)^2+a^2}\right\}^2 - \left\{\sqrt{x^2+a^2}\right\}^2}{\delta x\left(\sqrt{(x+\delta x)^2+a^2} + \sqrt{x^2+a^2}\right)}$$

$$= \frac{(x+\delta x)^2 + a^2 - (x^2+a^2)}{\delta x\left(\sqrt{(x+\delta x)^2+a^2} + \sqrt{x^2+a^2}\right)}$$

$$\frac{\partial y}{\partial x} = \frac{(x+\partial x)^2 + \cancel{a^2} - x^2 - \cancel{a^2}}{\partial x\left(\sqrt{(x+\partial x)^2+a^2} + \sqrt{x^2+a^2}\right)}$$

$$= \frac{\cancel{x^2} + 2x.\partial x + (\partial x)^2 - \cancel{x^2}}{\partial x\left(\sqrt{x^2+2x.\partial x+(\partial x)^2+a^2} + \sqrt{x^2+a^2}\right)}$$

$$= \frac{\partial x(2x+\partial x)}{\partial x\left(\sqrt{x^2+2x.\partial x+(\partial x)^2+a^2} + \sqrt{x^2+a^2}\right)}$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{2x+\partial x}{\sqrt{x^2+2x.\partial x+(\partial x)^2+a^2} + \sqrt{x^2+a^2}}$$

$$\therefore \lim_{\partial x\to 0}\frac{\partial y}{\partial x} = \lim_{\partial x\to 0}\frac{2x+\partial x}{\sqrt{x^2+2x.\partial x+(\partial x)^2+a^2} + \sqrt{x^2+a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+0}{\sqrt{x^2+2x.0+0^2+a^2} + \sqrt{x^2+a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{\sqrt{x^2+a^2} + \sqrt{x^2+a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{2}x}{\cancel{2}.\sqrt{x^2+a^2}}$$

$$\therefore \frac{dy}{dx} = \frac{x}{\sqrt{x^2+a^2}}$$

Ex: Find the derivative of x^n by using the first Principle

Solⁿ: Let $y = x^n \longrightarrow (i)$

$\therefore y+\partial y = (x+\partial x)^n \longrightarrow (ii)$

$\therefore (ii)-(i) \Rightarrow (y+\partial y) - y = (x+\partial x)^n - x^n$

$$\Rightarrow \partial y = \left[\cancel{x^n} + n.x^{n-1}.\partial x + \frac{n(n-1)}{2!}x^{n-2}.(\partial x)^2 + \cdots\right] - \cancel{x^n}$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{n.x^{n-1}.\partial x + \frac{n(n-1)}{2!}.x^{n-2}.(\partial x)^2 + \cdots}{\partial x}$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{\partial x\left[n.x^{n-1} + \frac{n(n-1)}{2!}.x^{n-2}.(\partial x) + \cdots\right]}{\partial x}$$

$$\Rightarrow \frac{\partial y}{\partial x} = n.x^{n-1} + \frac{n(n-1)}{2!}.x^{n-2}.(\partial x) + \cdots$$

$$\therefore \lim_{\partial x \to 0} \frac{\partial y}{\partial x} = \lim_{\partial x \to 0} \left[n.x^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot (\partial x) + \cdots \right]$$

$$\Rightarrow \frac{dy}{dx} = n.x^{n-1}$$

§ A few Theorems on differentiation:

a) Derivative of a constant

Let $y = f(x) = c$, where c is a constant

$y + \partial y = f(x+\partial x) = c$ [$\because$ $f(x)$ doesnot change with the change in x]

$$\therefore y + \partial y - y = f(x+\partial x) - f(x)$$

$$\Rightarrow \partial y = f(x+\partial x) - f(x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{f(x+\partial x) - f(x)}{\partial x}$$

$$\Rightarrow \lim_{\partial x \to 0} \frac{\partial y}{\partial x} = \lim_{\partial x \to 0} \frac{f(x+\partial x) + f(x)}{\partial x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\partial x \to 0} \frac{c-c}{\partial x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\partial x} (0)$$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\therefore \frac{d}{dx}(c) = 0 \text{; where } c \text{ is a constant}$$

b) The derivative of the product of a constant and a function.

Let $y = c.f(x)$; where c is a constant

$$\therefore y + \partial y = c.f(x+\partial x)$$

$$y + \partial y - y = c.f(x+\partial x) - c.f(x)$$

$$\Rightarrow \partial y = c.\left[f(x+\partial x) - f(x)\right]$$

$$\Rightarrow \frac{\partial y}{\partial x} = c.\frac{f(x+\partial x) - f(x)}{\partial x}$$

$$\Rightarrow \lim_{\partial x \to 0} \frac{\partial y}{\partial x} = c\left[\lim_{\partial x \to 0} \frac{f(x+\partial x) - f(x)}{\partial x}\right]$$

$$\Rightarrow \frac{dy}{dx} = c.\frac{d}{dx}\left[f(x)\right]$$

$$\Rightarrow \frac{d}{dx}\left[c.f(x)\right] = c.\frac{d}{dx}\left[f(x)\right]$$